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INFINITE GAMES WITH IMPERFECT INFORMATION(1)

BY

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ABSTRACT. We consider an infinite, two person zero sum game played as follows: On the nth move, players A, B select privately from fixed finite sets, A_n, B_n, the result of their selections being made known before the next selection is made. After an infinite number of selections, a point in the associated sequence space, Ω, is produced upon which B pays A an amount determined by a payoff function defined on Ω. In this paper we extend a result of Blackwell and show that if the payoff function is the indicator function of a set in the Boolean algebra generated by the G_δ's (with respect to a natural topology on Ω) then the game in question has a value.

1. **Introduction.** Infinite games with imperfect information have been studied by several writers, notably Blackwell [1], [2], and Shapley [5]. Before proceeding with the main result of this paper, we will discuss the structure and admissible strategies of these games.

Let $\{A_n\}, \{B_n\}$ be sequences of nonempty finite sets. Let $Z_n = A_n \times B_n$, and let Ω be the space $\prod_{n=1}^{\infty} Z_n$ of infinite sequences $\omega = (z_1, z_2, \cdots)$ where $z_n \in Z_n$. Let Ω be topologized as follows (for a related discussion, see [3]):

Suppose X is the set of all positions, i.e. finite sequences, $x = (z_1, z_2, \cdots, z_n)$, $z_i \in Z_i$, $n = 0, 1, 2, \cdots$. Then if $\omega \in \Omega$, $x \in X$, we define x to be a neighborhood of ω if ω passes through x. If the positions are thus considered as sets, they form a base for a Hausdorff, disconnected topology for Ω in which Ω is compact.

In this topology any open set is defined by a subset of X (a countable collection of positions). Any set defined by a finite collection of positions is both open and closed. It is shown by Wolfe [7] that if G is a G_δ then there exists a collection of positions T such that $G = \{\omega \in \Omega \mid \omega$ passes through infinitely many members of $T\}$, which we will henceforth denote by $G = T$ i.o.

Now, suppose f is a bounded Baire function on Ω. Then we define a zero sum two person game G_f, played as follows:

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First, player A selects $a_1 \in A_1$ while player B simultaneously selects $b_1 \in B_1$. The result, $z_1 = (a_1, b_1) \in Z_1$, is announced to both players, upon which A selects $a_2 \in A_2$ while B selects $b_2 \in B_2$, etc. The result of this infinite sequence of moves is a point $\omega = (z_1, z_2, \cdots) \in \Omega$ and B pays A the amount $f(\omega)$.

A strategy α (β) for A (B) gives for each position x (of length n, say) a probability distribution on A_{n+1} (B_{n+1}) with the stipulation that if the current position is x, A (B) will make his next choice according to α (β). A pair of strategies (α, β) defines a probability distribution $P_{\alpha\beta}$ on Ω and, hence, an expected payoff to A in G_f when A uses α and B uses β:

$$E(f, \alpha, \beta) = \int f(\omega)\, dP_{\alpha\beta}(\omega).$$

The lower and upper values of G_f are, respectively,

$$L(G_f) = \sup_{\alpha} \inf_{\beta} E(f, \alpha, \beta), \qquad U(G_f) = \inf_{\beta} \sup_{\alpha} E(f, \alpha, \beta).$$

It is always true that $L(G_f) \le U(G_f)$; if $L(G_f) = U(G_f)$, this common value is called the value of G_f and will be denoted by $\mathrm{Val}(G_f)$.

2. **Main Result.** We will show that if $f = I_G$, where $G \in B(G_\delta)$ (the Boolean algebra generated by the G_δ's), then G_f has a value. In [1], Blackwell proved this result if G is a G_δ. Before proving this result, we give two examples of games of this type and mention a related open question.

Example 1. On each move, players A and B choose, simultaneously, a 0 or 1. The winning set S of the form $G_\delta \cup F_\sigma$, is defined as follows:

$S = G \cup F$ where $G = \{\omega \mid \omega_n = (0, 0)$ for infinitely many n and $\omega_n = (1, 1)$ for infinitely many $n\}$,

$F = \{\omega \mid \omega_n = (0, 0)$ for at most finitely many n and $\omega_n = (1, 1)$ for at most finitely many $n\}$.

The value of this game is 1, which can be achieved by A with a nonrandom strategy; he starts by saying 1 on each move. If B says 0 on every move, F is hit. If B ever says 1, A then starts saying 0's. If B then says 1's forever, F is still hit. If B ever says 0 again, A switches back to 1's, etc. If there are an infinite number of changes G is hit, otherwise F is hit.

Example 2. The winning set is a G_δ. On each move, the players choose simultaneously a 0 or 1. If player A ever says 1, the game is over on that move; if B also said 1, A wins; if B said 0, B wins. If A never says 1, the game continues and A wins if there are infinitely many moves with outcome (0, 0). (In other words, A tries to predict B's choice. See [2] for a related game.)

The value of this game is 1, but there are no optimal strategies for A. Here is a strategy for A, due to David Blackwell, which, for fixed N, guarantees A at least $1 - 1/N$:

Define $N_j = 2^j N$, $j = 1, 2, \cdots$, so that

$$\sum_{j=1}^{\infty} \frac{1}{N_j} = \frac{1}{N}.$$

Player A divides the trials into successive blocks of lengths $N_1, N_2, \cdots$. If he has not yet stopped the game, i.e. played 1 when block j is reached, he selects X_j at random from $\{1, 2, \cdots, N_j\}$. He then plays 1 at the X_jth trial of block j if B's previous $X_j - 1$ plays in the block are all 1's; otherwise, he plays 0 throughout the block. Then, clearly,

$$P(A \text{ loses on } j\text{th block} \mid j\text{th block is reached}) \leq 1/N_j.$$

Thus, $P(A$ loses on jth block$) \leq 1/N_j$, and $P(A$ loses by failing to match$) \leq \Sigma(1/N_j) = 1/N$. However, by the nature of this strategy, if the game goes on forever, A will win, since there would then be (0, 0)'s in each block. □

The following question remains unsolved in general: Do games with payoffs which are simple functions based on sets in $B(G_\delta)$ have a value, i.e. games with payoff of the form $f = c_1 I_{B_1} + \cdots + c_n I_{B_n}$, where $B_i \in B(G_\delta)$, c_i are constants. In fact, we do not even know whether or not the much simpler games with payoffs of the form $I_{Q_1} - I_{Q_2}$ have a value, where Q_1, Q_2 are open and disjoint. In another paper we will discuss some special cases of these kinds of games and show that they have a value.

We are now ready to prove the main result.

Lemma 1. *Consider the class of sets of the form*

$$G_1 \cup F_1 \cup (G_2 \cap F_2) \cup \cdots \cup (G_n \cap F_n)$$

where $G_i \in G_\delta$, $F_i \in F_\sigma$. This class of sets is precisely $B(G_\delta)$.

Proof. By the fact that a finite union of G_δ's is a G_δ, a finite intersection of F_σ's is an F_σ, and by the standard results for generating Boolean algebras (e.g. see [4, Proposition I. 2.2, p. 7]) it is easily shown that every set in $B(G_\delta)$ is of the form $\bigcup_{i=1}^n (G_i \cap F_i)$, $G_i \in G_\delta$, $F_i \in F_\sigma$. Thus, every set in $B(G_\delta)$ is of the form

$$\left(\bigcup_{i=1}^{n} (G_i \cap F_i)\right)^c = \bigcap_{i=1}^{n} (G_i^c \cup F_i^c)$$

$$= \left(\bigcap_{i=1}^{n} F_i^c\right) \cup \left(\bigcap_{i=1}^{n} G_i^c\right) \cup \left(\bigcup_{i=1}^{n} \left(F_i^c \cap \left(\bigcap_{j \neq i} G_j^c\right)\right)\right).$$

which, again using the fact about finite unions (intersections) of G_δ's (F_σ's) is easily seen to be of the required form.

Lemma 2. *Consider the class of sets* $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ *generated in the following way:* $\mathcal{C}_1 = G_\delta$; *if* $n > 1$, $\mathcal{C}_n$ = *sets of the form* $G_\delta \cup A_{n-1}$, *where* A_n = *complements of sets in* $\mathcal{C}_n$ (*e.g.* $\mathcal{C}_2 = G_\delta \cup F_\sigma$). *We claim* $\mathcal{C} = B(G_\delta)$.

Proof. By using De Morgan's law and the fact that $G_\delta \cup G_\delta = G_\delta$, $F_\sigma \cap F_\sigma = F_\sigma$, it is easily seen that $\mathcal{C}_4$ = sets of the form $G_1 \cup F_1 \cup (F_2 \cap G_2)$, $\mathcal{C}_{2n}$ = sets of the form $G_1 \cup F_1 \cup (\bigcup_{i=2}^{n} (F_i \cap G_i))$ which by Lemma 1 gives the sets in $B(G_\delta)$. □

The next lemma is the first step in an induction which will yield the main result.

Lemma 3. *Let* ϕ *be upper semicontinuous,* $0 \le \phi \le 1$. *Suppose* $G \in G_\delta$. *Then the game with payoff* $\bar{\phi} = \min(\phi, I_G)$ *has a value.*

Proof. The first part of this proof and A's method of play is the same as in [1]. Suppose $G = T$ i.o., where T is a collection of positions. For any position x, let G_x^* be the game, starting from x with payoff $U_t(G_{\bar{\phi}})$ if T is hit for the first time after x at t, with payoff 0 if T is never hit after x, where $U_t(G_{\bar{\phi}})$ is the upper value of the original game starting from t. This payoff is lower semicontinuous, so, by [6], G_x^* has a value and player B has an optimal strategy. We claim $\mathrm{Val}(G_x^*) \ge U_x(G_{\bar{\phi}})$; for fixed $\epsilon > 0$, we present a strategy for B starting from x such that no matter what A does, $E_x(\bar{\phi}) \le \mathrm{Val}(G_x^*) + \epsilon$:

Let B, starting from x, play optimally in G_x^* until T is hit for the first time after x, say at t. Then B plays, starting from t, to keep $E_t(\bar{\phi}) \le U_t(G_{\bar{\phi}}) + \epsilon$, so

$$E_x(\bar{\phi}) = \sum_{t \in T} p(t) E_t(\bar{\phi}) \le \sum_{t \in T} p(t) U_t(G_{\bar{\phi}}) + \epsilon \le \mathrm{Val}(G_x^*) + \epsilon.$$

Now, for $\epsilon > 0$, we describe a strategy for A such that no matter what B does, $E(\bar{\phi}) \ge U(G_{\bar{\phi}}) - \epsilon$, and the lemma will be proved. First, A plays $\epsilon/4$ optimally in G_e^* (e denotes the empty position). If T is hit after e, say at t_1, A then plays $\epsilon/8$ optimally in $G_{t_1}^*$, etc. (If T is hit for the nth time at t_n, A then plays $\epsilon/2^{n+1}$ optimally in $G_{t_n}^*$.) Let the resulting sequence of moves be denoted by $z = (z_1, z_2, \cdots)$.

We define a sequence of random variables: $X_0 = U(G_{\bar{\phi}})$; for $k \ge 1$, $X_k = U_{t_k}(G_{\bar{\phi}})$ if T was hit for the kth time at t_k, $X_k = 0$ if T was hit less than k times. Thus, we have

$$E(X_k \mid X_{k-1}, \cdots, X_0) \ge X_{k-1} - \epsilon/2^{k+1}. \tag{1}$$

This is obvious if $X_{k-1} = 0$. If not, T was hit for the $(k-1)$st time at t_{k-1}, say, after which A played $\epsilon/2^{k+1}$ optimally in $G^*_{t_k}$ to get at least $\mathrm{Val}(G^*_{t_k}) - \epsilon/2^{k+1} \geq X_{k-1} - \epsilon/2^{k+1}$. Since the payoff in $G^*_{t_k}$ is X_k, (1) follows. Taking expectations on both sides, we get

$$E(X_k) \geq E(X_{k-1}) - \epsilon/2^{k+1} \Rightarrow E(X_k) \geq U(G_{\bar{\phi}}) - \epsilon/2. \tag{2}$$

Now, by the definition of upper semicontinuity and the nature of the topology on Ω, for every point $z = (z_1, z_2, \cdots)$ and every $\epsilon > 0$, there exists k such that any point $\omega = (\omega_1, \omega_2, \cdots)$ with $\omega_i = z_i$ for $i \leq k$ has the property

$$\begin{aligned} \phi(\omega) \leq \phi(z) + \epsilon/2 &\Rightarrow \text{if } z \in G,\ \bar{\phi}(\omega) \leq \bar{\phi}(z) + \epsilon/2 \\ &\Rightarrow (\text{still if } z \in G)\ U_{(z_1, \cdots, z_k)}(G_{\bar{\phi}}) \leq \bar{\phi}(z) + \epsilon/2 \\ &\Rightarrow (\text{for any } z) \limsup_n X_n \leq \bar{\phi}(z) + \epsilon/2. \end{aligned}$$

The last implication is obvious if $z \in G$. If $z \notin G$, T is only hit say N times, so for $n \geq N$, $X_n = 0$. Using Fatou's lemma on the last inequality, we get

$$E(\bar{\phi}) \geq \limsup_n E(X_n) - \epsilon/2 \geq U(G_{\bar{\phi}}) - \epsilon. \quad \square$$

Theorem 1. *Suppose $H \in \mathcal{C}_n$, i.e. $H = G \cup S$, $G \in G_\delta$, $S^c \in \mathcal{C}_{n-1}$. Suppose, also, that ϕ is upper semicontinuous with the property $0 \leq \phi \leq 1$, $\phi = 1$ on S. Then the game with payoff $\bar{\phi} = \min(\phi, I_H)$ has a value.*

Proof. Lemma 3 shows that the theorem is true for sets in $\mathcal{C}_1$. Suppose the theorem is true for sets in $\mathcal{C}_{n-1}$. Let $H \in \mathcal{C}_n$, $H = G \cup S$, where $G \in G_\delta$, $S^c \in \mathcal{C}_{n-1}$ (assume, without loss of generality, that $G \neq \emptyset$). Suppose $G = T$ i.o. for some collection of positions T. For any position x, let H^*_x be the game starting at x which continues until the first time T is hit after x, say at t, with A getting $U_t(G_{\bar{\phi}})$ when this happens. Otherwise, the game continues and A gets I_S. We claim H^*_x has a value (the payoff in H^*_x may be neither upper nor lower semicontinuous).

Observe that if C is closed, $J \in G_\delta \Rightarrow C \cap J \in G_\delta$; $J \in F_\sigma \Rightarrow C \cap J \in F_\sigma$; therefore $J \in \mathcal{C}_n \Rightarrow C \cap J \in \mathcal{C}_n$. Let $\mathcal{O}_x$ be the open set defined by the collection of positions passing through x which later hit T. Let $C_x = \mathcal{O}^c_x$. Then $S^c \cap C_x \in \mathcal{C}_{n-1}$ since S^c is. Define $\phi^* = 1 - f$, where f is the payoff in H^*_x. Also, define the upper semicontinuous function g: $g = \phi^*$ on $\mathcal{O}_x$, $g \equiv 1$ elsewhere. Thus g satisfies the conditions of the theorem and $\phi^* = \min(g, I_{C_x \cap S^c})$, so by the induction hypothesis the game starting at x with payoff ϕ^* has a value. Therefore, H^*_x has a value since its payoff is $f = 1 - \phi^*$ (the method of proof in Lemma 3 allows negation of the payoff since the same proof can be used by reversing the role of the players; it clearly allows the addition of a constant to the payoff).

By reasoning similar to that in Lemma 3, it can be shown that $\mathrm{Val}(H_x^*) \geq U_x(G_{\bar\phi})$. Now, for fixed $\epsilon > 0$, we will exhibit a strategy for player A which guarantees that $E(\bar\phi) \geq U(G_{\bar\phi}) - \epsilon$.

As in Lemma 3, A starts out playing $\epsilon/4$ optimally in H_e^*, etc. (If T is hit for the nth time at t_n, A then plays $\epsilon/2^{n+1}$ optimally in $H_{t_n}^*$.) Let the resulting play be $z = (z_1, z_2, \cdots)$.

Define the random variables: $X_0 = U(G_{\bar\phi})$; for $k \geq 1$, $X_k = U_{t_k}(G_{\bar\phi})$ if T is hit for the kth time at t_k, $X_k = I_S$ if T is hit less than k times. By reasoning similar to that in Lemma 3, we get

$$E(X_k) \geq U(G_{\bar\phi}) - \epsilon/2. \tag{3}$$

Again, by the definition of upper semicontinuity, there exists $k_{(\epsilon,z)}$ such that any point $\omega = (\omega_1, \omega_2, \cdots)$ agreeing with z up to $z_{k(\epsilon,z)}$ has the property

$$\begin{aligned}\phi(\omega) \leq \phi(z) + \epsilon/2 &\Rightarrow \text{if } z \in H,\ \bar\phi(\omega) \leq \bar\phi(z) + \epsilon/2\\ &\Rightarrow \text{if } z \in H,\ U_{(z_1,\cdots,z_k)}(G_{\bar\phi}) \leq \bar\phi(z) + \epsilon/2\\ &\Rightarrow \text{for any } z,\ \limsup_n X_n \leq \bar\phi(z) + \epsilon/2.\end{aligned}$$

Again, the last step is obvious if $z \in G$. If not, $S \cap G^c$ is hit and $\bar\phi = 1$ or $S^c \cap G^c$ is hit and $\limsup_n X_n = 0$. Thus, by Fatou, $E(\bar\phi) \geq \limsup_n E(X_n) \geq U(G_{\bar\phi}) - \epsilon$. □

Corollary 1. *If $H \in B(G_\delta)$, the game with payoff I_H has a value.*

Proof. Let $\phi = 1$ and use the theorem and Lemma 2.

Corollary 2. *G_f has a value if f satisfies the following conditions*:

(a) *There is a collection T of nonoverlapping positions (nonoverlapping means $x \in T \Rightarrow x$ is not an initial segment of any other member of T) such that if $x \in T$ then f is constant on all sequences passing through x.*

(b) $0 \leq f \leq 1$ *on T.*

(c) *There exists $H \in B(G_\delta)$ such that $f = I_H$ if T is never hit.*

Proof. The function $\phi = 1$ off T, $\phi = f$ otherwise, is upper semicontinuous, $\phi = 1$ on H, and $f = \min(\phi, I_H)$ so the theorem applies. □

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