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MATHEMATICS
(MEASURE AND INTEGRATION)

A Blackwell Space Which Is Not Analytic

by

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Presented by K. KURATOWSKI on July 31, 1971

Summary. Blackwell proved in 1954 that if A is an analytic set (contained in a Polish space) and $\mathfrak{F}$ is arbitrary countably generated σ-field contained in σ-field of $\mathcal{B}_A$ of the relatively Borel subsets A and such that all singletons belong to $\mathfrak{F}$, then $\mathfrak{F}=\mathcal{B}_A$. In the paper the author proves that in every uncountable Polish space there is contained a set A having the above property of Blackwell and such that neither A nor the complement of A contains uncountable analytical set.

A countably generated σ-field $\mathfrak{F}$ of subsets of a set X is called a Blackwell space if for every countably generated σ-field $\mathcal{C}\subset\mathfrak{F}$ having the same atoms as $\mathfrak{F}$, then $\mathcal{C}=\mathfrak{F}$. An equivalent characterization is the following: If f is a 1-1 Borel measurable function from X into a countably generated measurable space, then f^{-1} is also Borel (e.g., see [3]). In [1], Blackwell proved that every analytic subset of a Polish space (endowed with the relative Borel σ-field) is a Blackwell space. The converse to this result has remained an open question. In this paper we prove the converse to be false; if X is a Polish space (uncountable complete separable metric space), we construct a non-analytic (in fact, non measurable) subset of X which is a Blackwell space.

The Construction. Let X be an (uncountable) Polish space. Let $\mathcal{B}$ be the σ-field of Borel sets of X. Using transfinite induction, we will construct a set $A\subset X$ with the following properties:

1* *Neither A nor A^C contains an uncountable analytic set (thus, A is not analytic nor measurable with respect to the completion of any measure on $\mathcal{B}$).*

2* *If f is a Borel function from $X\to X$ such that $S_f=\{x\mid \operatorname{card} f^{-1}(x)>1\}$ is uncountable, then there exists distinct x, y in A such that $f(x)=f(y)$.*

We proceed with the construction; we first well order the class of uncountable Borel sets in X (This class has power c). We next well order the class of Borel functions $f\colon X\to X$ which satisfy the condition that S_f is uncountable. This class of functions also has power c. We inductively construct two disjoint collections of nested sets, A_α, E_α, as follows:

First, we select four distinct members of X, a_1, b_1, c_1, d_1, where a_1, d_1 are members of B_1, and $f_1(b_1)=f_1(c_1)$, and where B_1, f_1 are the first members of the previous orderings. We let $\{a_1, b_1, c_1\} = A_1$, $\{d_1\} = E_1$. When B_α, f_α are reached in the induction (where a is an ordinal less than c) we select four distinct points $a_\alpha, b_\alpha, c_\alpha, d_\alpha$ which have not previously been selected and where a_α, d_α are members of B_α and where $f_\alpha(b_\alpha) = f_\alpha(c_\alpha)$. We can do this because the sets B_α and S_{f_α} are resepctively Borel and analytic and since uncountable, must have power c. We then let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta \cup \cup \{a_\alpha, b_\alpha, c_\alpha\}$, $E_\alpha = \bigcup_{\beta<\alpha} E_\beta \cup \{d_\alpha\}$, etc. We then let $A = \bigcup_{\alpha<c} A_\alpha$. It is easily seen that A satisfies properties 1*, 2*.

We now consider the pair $(A, \mathcal{B}^0)$, where $\mathcal{B}^0$ is the relative Borel σ-field on A. Suppose that $g^0: A \to X$ is $\mathcal{B}^0$ measurable and 1-1. It is known (see [3], p. 434, VI) that g^0 can be extended to a $\mathcal{B}$ measurable function g, on all of X. We have, by the properties of A, that g has the following properties:

3* *All sets $g^{-1}(y)$ are countable (follows from 1*).*

4* *S_g is countable (otherwise, by 2*, for some distinct $u, v \in A$, $g^0(u) = g(u) = = g(v) = g^0(v)$, which cannot happen, since g^0 is 1-1).*

But 3* and 4* imply there is a countable set $N \in X$ such that the function $h = g$ restricted to $X \setminus N$ is 1-1. Since $X \setminus N$ is Borel, the inverse h^{-1} is a Borel function, thus, so is the inverse $g^{0\,-1}$, which completes the proof.

We wish to thank the referee, Professor C. Ryll-Nardzewski, for suggesting a simplified version of the original proof.

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М. Оркин, **Пространство Блекуэлла, которое не является аналитическим**

Содержание. Д. Блекуэлл доказал еще в 1954 году. что если A — аналитическое множество (содержащееся в польском пространстве) и если $\mathfrak{F}$ — произвольное счетно порождаемое σ-поле, содержащееся в σ-поле $\mathcal{B}_A$ релятивно борелевых подмножеств A и такое, что все сингельтоны принадлежат к $\mathfrak{F}$, то $\mathfrak{F} = \mathcal{B}_A$. В настоящей работе автор доказывает, что в каждом несчетном польском пространстве содержится множество A, обладающее вышеупомянутым свойством Блекуэлла и такое, что ни это множество, ни его дополнение не содержат несчетного аналитического множества.