AMERICAN MATHEMATICAL SOCIETY
www.ams.org

An Approximation Theorem for Infinite Games
Author(s): Michael Orkin
Source: *Proceedings of the American Mathematical Society*, Vol. 36, No. 1 (Nov., 1972), pp. 212-216
Published by: American Mathematical Society
Stable URL: http://www.jstor.org/stable/2039062
Accessed: 16-02-2018 17:34 UTC

REFERENCES
Linked references are available on JSTOR for this article:
http://www.jstor.org/stable/2039062?seq=1&cid=pdf-reference#references_tab_contents
You may need to log in to JSTOR to access the linked references.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
http://about.jstor.org/terms

JSTOR

American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the American Mathematical Society*

This content downloaded from 134.154.190.2 on Fri, 16 Feb 2018 17:34:21 UTC
All use subject to http://about.jstor.org/terms

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

AN APPROXIMATION THEOREM FOR INFINITE GAMES[1]

MICHAEL ORKIN

ABSTRACT. We consider infinite, two person zero sum games played as follows: On the nth move, players A, B select privately from fixed finite sets, A_n, B_n, the result of their selections being made known before the next selection is made. A point in the associated sequence space $\Omega=\prod_{n=1}^{\infty}(A_n\times B_n)$ is thus produced upon which B pays A an amount determined by a payoff function defined on Ω. We show that if the payoff functions of games $\{G_n\}$ are upper semicontinuous and decrease pointwise to a function which is the payoff for a game, G, then $\mathrm{Val}(G_n)\downarrow\mathrm{Val}(G)$. This shows that a certain class of games can be approximated by finite games. We then give a counterexample to possibly more general approximation theorems by displaying a sequence of games $\{G_n\}$ with upper semicontinuous payoff functions which increase to the payoff of a game G, and where $\mathrm{Val}(G_n)=0$ for all n but $\mathrm{Val}(G)=1$.

Introduction. Infinite games with imperfect information have been studied by several writers, notably Blackwell [**1**], [**2**], Gillette [**3**], Milnor and Shapley [**4**].

Before proceeding with the main result we will introduce notation and describe the structure of these games.

Let $\{A_n\}$, $\{B_n\}$ be sequences of nonempty finite sets. Let $Z_n=A_n\times B_n$ and let Ω be the space $\prod_{n=1}^{\infty}Z_n$ of infinite sequences $\omega=(z_1, z_2, \cdots)$ where $z_n\in Z_n$. Let $X=\{(z_1, z_2, \cdots, z_n)\,|\,z_i\in Z_i,\ n=1, 2, \cdots\}$ be the set of finite starting sequences or partial histories.

Suppose f is a bounded Baire function on Ω (with respect to the product topology). Then f, called a payoff function, defines a zero-sum two person game G_f, played as follows:

First, player A selects $a_1\in A_1$ while player B simultaneously selects

Received by the editors September 29, 1971.

AMS 1970 *subject classifications*. Primary 90D05; Secondary 60G45.

Key words and phrases. Imperfect information, zero sum two person game, infinite game, random strategy.

[1] This paper is part of the author's doctoral dissertation at the University of California, Berkeley, and was prepared with the partial support of U.S. Air Force Grant AF-AFOSR-1312-67 and U.S. Office of Naval Research Contract NONR N00014-66-C0036.

© American Mathematical Society 1972

This content downloaded from 134.154.190.2 on Fri, 16 Feb 2018 17:34:21 UTC
All use subject to http://about.jstor.org/terms

$b_1 \in B_1$. The result, $z_1=(a_1, b_1) \in Z_1$, is announced to both players, upon which A selects $a_2 \in A_2$ while B is selecting $b_2 \in B_2$, etc. The result of this infinite sequence of moves is a point $\omega = (z_1, z_2, \cdots) \in \Omega$ and B pays A the amount $f(\omega)$. For any partial history $x \in X$ we can define a subgame of the original game (usually referred to as the original game, "starting from x") by having the players play as above except redefining the payoff function as $f_x(\omega)=f(x\omega)$.

A strategy α (β) for A (B) gives for each partial history x (of length n, say) a probability distribution on A_{n+1} (B_{n+1}) with the stipulation that if the current position is x, A (B) will make his next choice according to α (β). A pair of strategies, (α, β) defines a probability distribution, $P_{\alpha\beta}$ on Ω and, hence, an expected payoff to A in G_f when A uses α and B uses β:

$$E(f, \alpha, \beta) = \int f(\omega)\, dP_{\alpha\beta}(\omega).$$

(We will usually omit the α, β from the notation when it is clear what is happening.)

The lower and upper values of G_f are, respectively,

$$L(G_f) = \sup_\alpha \inf_\beta E(f, \alpha, \beta), \qquad U(G_f) = \inf_\beta \sup_\alpha E(f, \alpha, \beta).$$

It is always true that $L(G_f) \leqq U(G_f)$; if $L(G_f)=U(G_f)$, this common value is called the value of G_f and will be denoted by $\mathrm{Val}(G_f)$.

Finally, a payoff function f is called upper (lower) semicontinuous if $\omega_n \to \omega \Rightarrow \limsup_n f(\omega_n) \leqq f(\omega)$ ($\liminf_n f(\omega_n) \geqq f(\omega)$).

The result of [5] we will use is as follows. Let M be compact, N any space, f defined on $M \times N$ which is concave-convexlike. If $f(\mu, \nu)$ is u.s.c. in μ for each ν, then $\sup_\mu \inf_\nu f = \inf_\nu \sup_\mu f$. We show how to apply this to the present situation: The space of plays, Ω, and the set of strategies for each player gives rise to a product of compact spaces, $\Omega_A^* \times \Omega_B^*$, where $\Omega_A^* = \prod_{n=1}^\infty A_n^*$, $\Omega_B^* = \prod_{n=1}^\infty B_n^*$. We define A_n^*, B_n^* as follows: If α is a strategy for player A, the corresponding member of Ω_A^* is a sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$, where $\alpha_n \in A_n^*$ is a finite list of probabilities on A_n, one for each possible past history (the list is finite since the sets A_n, B_n, $n=1, 2, \cdots$, are finite). B_n^* is defined analogously. The corresponding product topology makes Ω_A^*, Ω_B^* compact. If f is a payoff on Ω, we get a corresponding payoff f^* on $\Omega_A^* \times \Omega_B^*$ by defining $f^*(\alpha, \rho)=E(f, \alpha, \beta)$. If f is u.s.c. on Ω, so is f^* on $\Omega_A^* \times \Omega_B^*$ (in the product topology); also f^* is linear, so [5] applies. It is easily seen that $\sup_\alpha \inf_\beta f^* = \inf_\beta \sup_\alpha f^*$ implies the game with payoff f has a value, so [5] gives us that games with u.s.c. payoffs have a value.

This content downloaded from 134.154.190.2 on Fri, 16 Feb 2018 17:34:21 UTC
All use subject to http://about.jstor.org/terms

We shall now prove the main result, namely

THEOREM 2.1. *Suppose G_{f_n} are games with upper semicontinuous payoff functions f_n, where the f_n are (pointwise) nonincreasing, $f_n \downarrow f$ (which is, therefore, also upper s.c.). Then* $\mathrm{Val}(G_f) = \lim_n \mathrm{Val}(G_{f_n})$.

We first prove a lemma.

LEMMA 2.1. *Suppose f_n, f are as above. For any partial history x, let $m_x = \lim_n \mathrm{Val}_x(G_{f_n})$, where $\mathrm{Val}_x(G_{f_n})$ means the value of the game with payoff f_n, starting from x. (The value of games with upper s.c. payoff function exists, by* [5].) *Let G_x^* be the one move game which starts at x and has payoff $g = m_y$ if y is the next position hit. Claim* $\mathrm{Val}(G_x^*) \geqq m_x$.

PROOF OF LEMMA. We will show by contradiction that for fixed $\varepsilon > 0$, A can play in G_x^* to guarantee that $E(g) \geqq m_x - \varepsilon$. Assume not; then for every strategy of A, player B can play to make $E(g) < m_x - \varepsilon$.

For each possible next position, y_i, $i = 1, 2, \cdots, k$, let f_{n_i} be such that $\mathrm{Val}_{y_i}(G_{f_{n_i}}) < m_{y_i} + \varepsilon/2$. Let $m = \max_i(n_i)$; so that for all i,

$$\mathrm{Val}_{y_i}(G_{f_m}) < m_{y_i} + \varepsilon/2. \tag{1}$$

Now for any fixed strategy of player A, let B play according to the assumption, to make $E(g) < m_x - \varepsilon$ and then play $\varepsilon/2$ optimally in G_{f_m} to make

$$\begin{aligned} E_x(f_m) &\leqq \sum_{i=1}^{k} p(y_i)\mathrm{Val}_{y_i}(G_{f_m}) + \varepsilon/2 < \sum_{i=1}^{k} p(y_i) m_{y_i} + \varepsilon \quad \text{(by (1))} \\ &= E(g) + \varepsilon < m_x \end{aligned}$$

(by assumption) which contradicts the fact that $m_x = \lim_n \mathrm{Val}_x(G_{f_n})$, and the lemma is proved. □

Now we are ready for the

PROOF OF THEOREM 2.1. We shall show that for fixed $\varepsilon > 0$, A can guarantee that $E(f) \geqq \lim_n \mathrm{Val}(G_{f_n}) - \varepsilon = m_e - \varepsilon$ (where e denotes the empty sequence). This will complete the proof, since $\{\mathrm{Val}(G_{f_n})\}$ is a nonincreasing sequence, and so $U(G_f) \leqq \lim_n \mathrm{Val}(G_{f_n})$.

First, let A play optimally in G_e^*, and then, if x_n is the position after the nth move, let A play optimally in $G_{x_n}^*$. Define the random variables $X_0 = m_e$; if $n \geqq 1$, $X_n = m_{x_n}$ if x_n is the position after the first n moves. By the lemma, we have $E(X_n | X_{n-1} \cdots X_0) \geqq X_{n-1} \Rightarrow$

$$E(X_n) \geqq m_e \tag{2}$$

for all n.

Now, using the usual facts about upper semicontinuity, for fixed k (if $z = (z_1, z_2, \cdots)$ is the resulting sequence of moves), there exists

This content downloaded from 134.154.190.2 on Fri, 16 Feb 2018 17:34:21 UTC
All use subject to http://about.jstor.org/terms

$N_{(k,z,\varepsilon)}$ such that if $n \geqq N_{(k,z,\varepsilon)}$, any sequence $\omega = (\omega_1, \omega_2, \cdots)$ agreeing with z up to the nth move has the property

$$\begin{aligned} f_k(\omega) < f_k(z) + \varepsilon &\Rightarrow \mathrm{Val}_{(z_1, z_2, \ldots, z_n)}(G_{f_k}) < f_k(z) + \varepsilon \\ &\Rightarrow m_{(z_1, z_2, \ldots, z_n)} < f_k(z) + \varepsilon \\ &\Rightarrow \text{for all } z, \quad \limsup_n X_n(z) < f_k(z) + \varepsilon \\ &\Rightarrow \text{(by Fatou)} \quad \limsup_n E(X_n) < E(f_k) + \varepsilon \\ &\Rightarrow E(f_k) > m_e - \varepsilon \quad \text{for all } k \\ &\Rightarrow E(f) > m_e - \varepsilon \end{aligned}$$

(by the dominated convergence theorem). □

COROLLARY 1. *If f_n are lower semicontinuous, $f_n \uparrow f$, then* $\lim_n \mathrm{Val}(G_{f_n}) = \mathrm{Val}(G_f)$.

PROOF. The negative of an u.s.c. function is l.s.c. so the theorem applies by reversing the roles of the players.

COROLLARY 2. *Games with lower semicontinuous payoff functions can be approximated by finite games.*

PROOF. Suppose f is l.s.c. Define f_n by $f_n(\nu) = \inf_{\omega \in S} f(\omega)$ where $S = \{\omega \in \Omega \mid 1\text{st } n \text{ coordinates of } \omega \text{ agree with the 1st } n \text{ coordinates of } \nu\}$. Then the games G_{f_n} are "finite", since the payoff is decided in the first n moves. But the fact that f is l.s.c. implies $f_n \uparrow f$, so we just apply Corollary 1. (The functions f_n are continuous.)

COROLLARY 3. *Open games can be approximated by finite games, i.e., if $f = I_{\mathcal{O}}$ where $\mathcal{O}$ is an open set* (*in the product topology on* Ω) *then the game G can be approximated by the games G_n, where the payoff in G_n is* 1 *if $\mathcal{O}$ is hit by the nth move*, 0 *otherwise*. (*This is actually a special case of Corollary* 2.)

PROOF. Immediate since $I_{\mathcal{O}}$ is l.s.c.

A COUNTEREXAMPLE. Approximation theorems do not exist in general as the following example shows. Let $A_n = B_n = \{0, 1\}$ for $n = 1, 2, \cdots$, so $\Omega = \prod_{n=1}^{\infty} \{0, 1\} \times \{0, 1\}$. Let $S_n = F_n \cup G$ where $F_n = \{\omega \in \Omega \mid \exists i \leqq n \text{ with } \omega_i = (1, 1)\}$ (in other words $F_n = \{\omega \mid \text{both players say 1 on the same move sometime before the } n\text{th move}\}$), and $G = \{\omega \in \Omega \mid \text{player } B \text{ says 0 on every move}\}$. Clearly F_n and G are closed sets, so the functions I_{S_n} are upper semicontinuous. Now the games G_n with payoffs I_{S_n} have value 0 since player B need only say 0 for the first n moves and 1 sometime after that to keep play from hitting S_n. Also since $S_{n+1} \supset S_n$ for all n, $I_{S_n} \uparrow I_S$ where $S = \bigcup_{n=1}^{\infty} S_n$. But the game with payoff I_S has value 1 which player A

This content downloaded from 134.154.190.2 on Fri, 16 Feb 2018 17:34:21 UTC
All use subject to http://about.jstor.org/terms

can achieve by merely saying 1 on every move. Player B either must say 0 every time or 1 sometime and so S is hit.

AN OPEN QUESTION. We do not know whether if f_n are continuous, $f_n \to f$, then Val $G(F_n) \to$ Val $G(F)$. This question has some relevance to the study of stochastic games (see [**2**], [**3**]).

I wish to express my gratitude to David Blackwell for many helpful conversations during the course of this research.

REFERENCES

1. D. Blackwell, *Infinite G_δ-games with imperfect information*, Zastos. Mat. **10** (1969), 99–101. MR **39** #5158.

2. D. Blackwell and T. S. Ferguson, *The big match*, Ann. Math. Statist. **39** (1968), 159–163. MR **36** #6211.

3. D. Gillette, *Stochastic games with zero stop probabilities*, Contributions to the Theory of Games, vol. 3, Ann. of Math. Studies, no. 39, Princeton Univ. Press, Princeton, N.J., 1957, pp. 179–187. MR **19**, 1147.

4. J. W. Milnor and L. S. Shapley, *On games of survival*, Contributions to the Theory of Games, vol. 3, Ann. of Math. Studies, no. 39, Princeton Univ. Press, Princeton, N.J., 1957, pp. 15–45. MR **19**, 1147.

5. M. Sion, *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176. MR **20** #3506.

DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106

This content downloaded from 134.154.190.2 on Fri, 16 Feb 2018 17:34:21 UTC
All use subject to http://about.jstor.org/terms